Permutations and Combinations

It’s easy to confuse “permutation” and “combination” — which one’s which?

Here’s an easy way to remember: **permutation sounds complicated**, doesn’t it? And it is. With permutations, every little detail matters. “Alice, Bob and Charlie” is different from “Charlie, Bob and Alice.”

Combinations, on the other hand, are pretty easy going. The details don’t matter. “Alice, Bob and Charlie” are the same as “Charlie, Bob and Alice.”

**Permutations are for lists (order matters) and combinations are for groups (order doesn’t matter).**

Permutations

The number of ways you can change the order of a set of things is called the number of **permutations** of that set of things.

Question: How many different ways can you arrange the letters in the word WHO?

Answer:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| WHO | WHO | HWO | HOW | OHW | OWH | = 6 ways |
| 1 | 2 | 3 | 4 | 5 | 6 |

Each different letter arrangement is called a permutation of the word "WHO". How about the word "STOP"?

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| STOP | STPO | SOTP | SOTP | SPTO | SPOT | ←starts with “S” |
| TSOP | TSPO | TOSP | TOPS | TPSO | TPOS | ←starts with “T” |
| OSTP | OSPT | OTSP | OTPS | OPST | OPTS | ←starts with “O” |
| PSTO | PSOT | PTSO | PTOS | POST | POTS | ←starts with “P” |

There are 24 ways to order the letters in "STOP". Is there a general rule here? Fortunately, yes. Here's the rule for "STOP":

There are 4 ways to pick the first letter.

After you pick the first letter there are 3 ways to pick the second letter.

After you pick the first 2 letters, there are 2 ways to pick the third letter.

After picking the first 3 letters, there is only 1 letter left to pick.

So the number of ways to order the letters in "STOP" is 4 · 3 · 2 · 1 = 24 ways!

Factorials

When you multiply a whole number by all the whole numbers below it, it’s called the **factorial** of that number. Factorials are used to compute permutations. In the previous example, we saw that 4 letters could be permuted 4 x 3 x 2 x 1 ways, or 24 ways. Six things can be permuted 6 x 5 x 4 x 3 x 2 x 1 ways or 720 ways. We write it in shorthand with an exclamation point, like this: 6! You can see from the table below that factorials get very large in a hurry:

|  |  |  |
| --- | --- | --- |
| 1! = |  | 1 |
| 2! = | 2 · 1 | 2 |
| 3! = | 3 · 2 · 1 | 6 |
| 4! = | 4 · 3 · 2 · 1 | 24 |
| 5! = | 5 · 4 · 3 · 2 · 1 | 120 |
| 6! = | 6 · 5 · 4 · 3 · 2 · 1 | 720 |
| 7! = | 7 · 6 · 5 · 4 · 3 · 2 · 1 | 5,040 |
| 8! = | 8 · 7 · 6 · 5 · 4 · 3 · 2 · 1 | 40,320 |
| 9! = | 9 · 8 · 7 · 6 · 5 · 4 · 3 · 2 · 1 | 362,880 |

*The number of permutations of n things is* ***n!***

Repeated Letters

There is a special case when the things you are ordering are not all different. This is the case when you are permuting the letters in the word "MOON". Here are the permutations:

|  |  |  |
| --- | --- | --- |
| MOON | MONO | MNOO |
| OMON | OMNO | OOMN |
| OONM | ONMO | ONOM |
| NOOM | NOMO | NMOO |

Why are there only 12, instead of 4 x 3 x 2 x 1 = 24 ways? Because you can't tell the "O"s apart, so their order doesn't matter. Suppose one of the "O"s was capital and the other one was small. Then you would have "MOoN" and "MoON". But our "O"s are alike and we can't tell the difference so we have "MOON" and "MOON", which are the same. We count them once. Since there are 2 "O"s, there are 2! = 2 x 1 = 2 ways of ordering them. So the number of permutations of letters in the word MOON is:

$\frac{4!}{2!}$ = $\frac{4 · 3 · 2 · 1}{2 · 1}$ = 4 · 3 = 12 ways

For the word "SASS", there are 4 letters and 3 "S"s so there are:

$\frac{4!}{3!}$ = $\frac{4 · 3 · 2 · 1}{3 · 2 · 1}$ = 4 ways of permuting the letters (SASS, SSSA, ASSS, SSAS)

*The number of permutations of n things where r of them are the same is:* $\frac{n!}{r!}$

More Permutations

What if we don’t want to use every item in a group? Let’s say we have 8 people:

1: Alice

2: Bob

3: Charlie

4: David

5: Eve

6: Frank

7: George

8: Horatio

How many ways can we pick a Gold, Silver, and Bronze medal for “Best friend in the world”?



We’re going to use permutations since the order we hand out these medals matters. Here’s how it breaks down:

* Gold medal: 8 choices: A B C D E F G H (Clever how I made the names match up with letters, eh?). Let’s say A wins the Gold.
* Silver medal: 7 choices: B C D E F G H. Let’s say B wins the silver.
* Bronze medal: 6 choices: C D E F G H. Let’s say… C wins the bronze.

We picked certain people to win, but the details don’t matter: we had 8 choices at first, then 7, then 6. The total number of options was 8 · 7 · 6 = 336.

Let’s look at the details. We had to order 3 people out of 8. To do this, we started with all options (8) then took them away one at a time (7, then 6) until we ran out of medals.

We know the factorial is: 8! = 8 · 7 · 6 · 5 · 4 · 3 · 2 · 1

Unfortunately, that does too much! We only want 8 · 7 · 6. How can we “stop” the factorial at 5?

This is where permutations get cool: notice how we want to get rid of 5 · 4 · 3 · 2 · 1. What’s another name for this? 5 factorial!

So, if we do $\frac{8!}{5!}$ we get:

$\frac{8 · 7 · 6 · 5 · 4 · 3 · 2 · 1}{5 · 4 · 3 · 2 · 1}$ = $\frac{8 · 7 · 6 · 5 · 4 · 3 · 2 · 1}{5 · 4 · 3 · 2 · 1}$ = 8 · 7 · 6

And why did we use the number 5? Because it was left over after we picked 3 medals from 8. So, a better way to write this would be:

$\frac{8!}{(8-3)!}$

which is just a fancy way of saying “Use the first 3 numbers of 8!”. If we have**n** items total and want to pick **k** in a certain order, we get:

$\frac{n!}{(n-k)!}$

which just means “Use the first k numbers of n!”

And this is the fancy permutation formula: You have **n** items and want to find the number of ways **k** items can be ordered:

P(n,k) = $\frac{n!}{(n-k)!}$

Combinations

Combinations are easy going. Order doesn’t matter. You can mix it up and it looks the same. Let’s say I’m a cheapskate and can’t afford separate Gold, Silver and Bronze medals. In fact, I can only afford empty tin cans.

How many ways can I give 3 tin cans to 8 people?

Well, in this case, the order we pick people doesn’t matter. If I give a can to Alice, Bob and then Charlie, it’s the same as giving to Charlie, Alice and then Bob. Either way, they’re going to be equally disappointed.

This raises an interesting point — we’ve got some redundancies here. Alice Bob Charlie = Charlie Bob Alice. For a moment, let’s just figure out how many ways we can rearrange 3 people.

Well, we have 3 choices for the first person, 2 for the second, and only 1 for the last. So we have 3 · 2 · 1 ways to re-arrange 3 people.

Wait a minute… this is looking a bit like a permutation! You tricked me!

Indeed I did. If you have N people and you want to know how many arrangements there are for **all** of them, it’s just N factorial or N!

So, if we have 3 tin cans to give away, there are 3! = 3 · 2 · 1 = 6 variations for every choice we pick. If we want to figure out how many combinations we have, we just **create all the permutations and divide by all the redundancies**. In our case, we get 336 permutations (from above), and we divide by the 6 redundancies for each permutation and get 336/6 = 56.

The general formula is:

C(n,k) = $\frac{P(n,k)}{k!}$

which means “Find all the ways to pick k people from n, and divide by the k! variants.” Writing this out, we get our **combination formula**, or the number of ways to combine k items from a set of n:

C(n,k) = $\frac{n!}{\left(n-k\right)!k!}$

Combinations vs. Permutations

Here’s an example of when we use combinations (order doesn’t matter) vs. permutations (order matters):

Combination: Picking a team of 3 people from a group of 10.

C(10,3) = $\frac{10!}{\left(10-3\right)!3!}$ = $\frac{10!}{7!3!}$ = $\frac{10 · 9 · 8 · 7 · 6 · 5 · 4 · 3 · 2 · 1}{(7 · 6 · 5 · 4 · 3 · 2 · 1)(3 · 2 · 1)}$ = $\frac{720}{6}$ = 120

Permutation: Picking a President, VP and Treasurer from a group of 10.

P(10,3) = $\frac{10!}{\left(10-3\right)!}$ = $\frac{10!}{7!}$ = $\frac{10 · 9 · 8 · 7 · 6 · 5 · 4 · 3 · 2 · 1}{(7 · 6 · 5 · 4 · 3 · 2 · 1)}$ = 720